# Lesson 31

# Other Geometries

Today, we'll briefly retrace the history of Geometry up to Euclid, and look at how it has changed since he wrote his ***Elements*** almost 2500 years ago (c. 400BC).

## Egyptians c. 2000 - 500 B.C.

Remember that the Egyptians new of practical geometric principles through surveying and construction projects.

- Each year when the Nile River overflowed, the land on its banks had to be re-surveyed.
- The "rope-stretchers" and pyramid builders knew about Pythagorean triples.
- They even approximated $\pi$, as it was found on the Rhind Papyrus.

## Babylonians c. 2000 - 500 B.C.

Ancient clay tablets reveal that they knew the Pythagorean relationships. One clay tablet reads

> *4 is the length and 5 the diagonal. What is the breadth? Its size is not known. 4 times 4 is 16. 5 times 5 is 25. You take 16 from 25 and there remains 9. What times what shall I take in order to get 9? 3 times 3 is 9. 3 is the breadth.*

## Greeks c. 750-250 B.C.

Ancient Greeks practiced centuries of experimental geometry like Egypt and Babylonia had, and they absorbed the experimental geometry of both of those cultures. Then they created the first formal mathematics of any kind by organizing geometry with rules of logic.

**Euclid's** *Elements* formed the basis for most of the geometry studied in schools ever since, including the topics on this show.

Let's review the two main types of mathematical (including geometric) rules :

## Postulates (also called axioms)

Postulates are basic assumptions - rules that seem to be obvious and are therefore accepted without proof.

## Theorems

Theorems are rules that must be proved.

Euclid gave five postulates. Everything went swimmingly until he got the fifth and final one. It reads:

> *If a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.*

For centuries, many mathematicians believed that this statement was not a true postulate, but rather a theorem which could be derived from the first four.

The fifth postulate came to be known as the ***parallel postulate*** since Scottish mathematician John Playfair (1748-1819) gave an acceptable equivalent rewording of Euclid's postulate. It is known as Playfair's axiom and reads:

*Given a line and a point not on the line, it is possible to draw exactly one line through the given point parallel to the line.*

Over the years, many purported proofs of the parallel postulate were published. However, none were correct, including the 28 "proofs" G. S. Kluegel analyzed in his dissertation of 1763 at the University of Goettingen.

This is where a remarkable mathematician enters the story. Carl Friedrich Gauss was came to the University of Goettingen in 1795 at age 18. He became interested in the postulate (among many other things).

Finally, in 1816 as a professor of mathematical astronomy at the same university, Gauss had the breakthrough that had been waiting since Euclid.

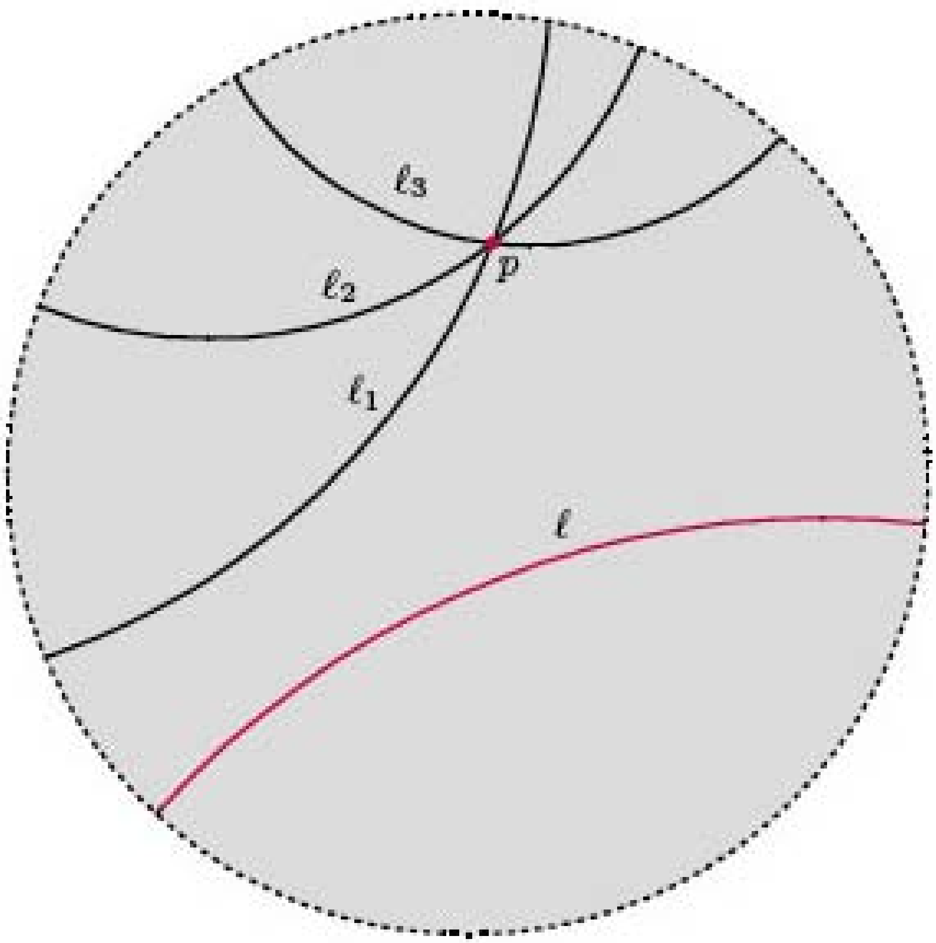
He wrote to a friend,

> The assumption that the sum of the three angles [of a triangle] is less than $180^\circ$ leads to a special geometry, quite different from ours [i.e. Euclidean], which is absolutely consistent, and which I have developed quite satisfactorily for myself . . ."

Although he did not publish his findings, he was successful in DISPROVING the parallel postulate, and in the process set the groundwork for a new geometry called **Hyperbolic Geometry**.

Later, Hungarian mathematician János Bolyai and Russian Nikolai Ivanovich Lobachevsky made the same independent discoveries, though none could visualize it.

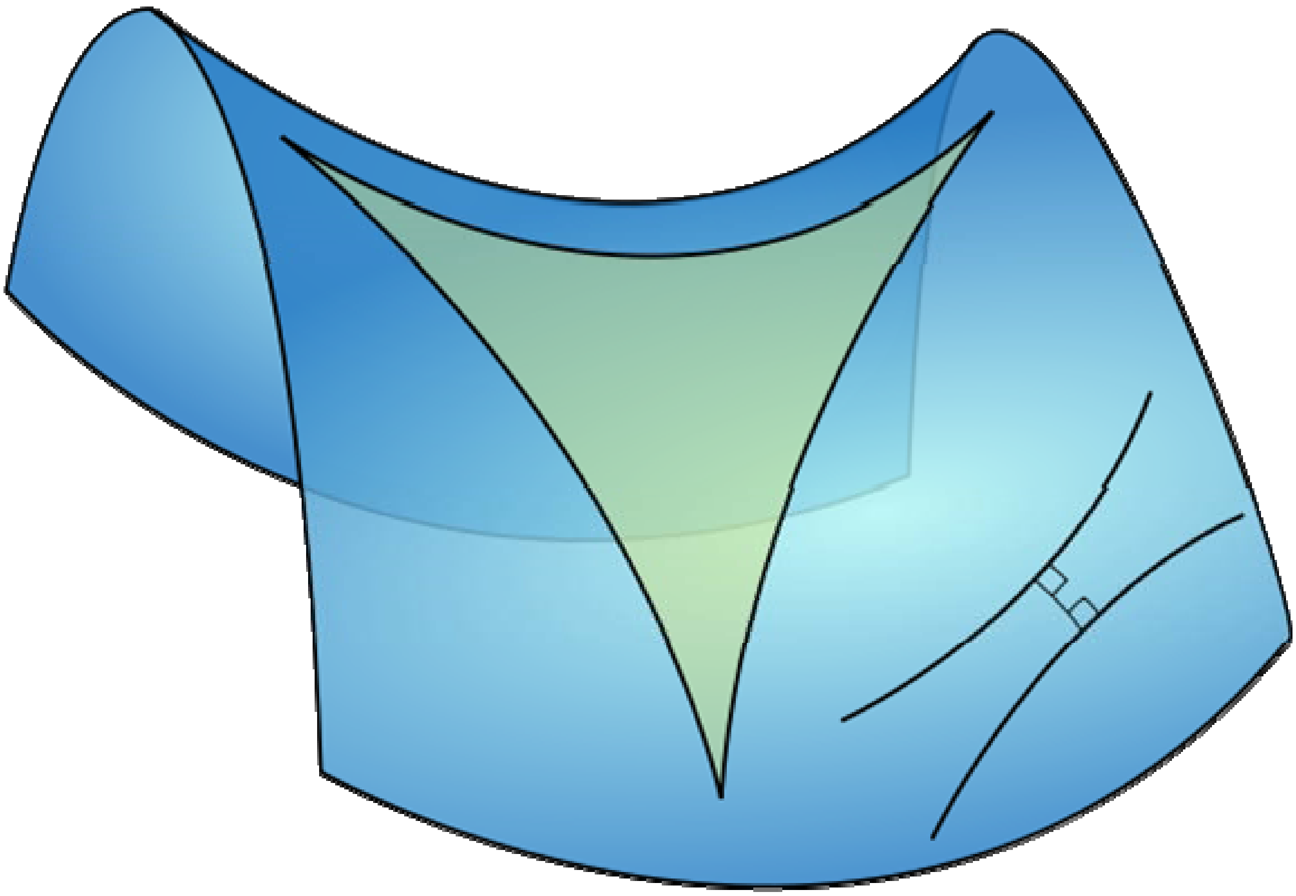
So what is Non-Euclidean Hyperbolic space? What does it look like? What are its consequences? If we take the parallel postulate and replace it with the assumption **that for any line, there are not just one, but MANY parallel lines through any given external point**, we get a hyperbolic geometry.




*This 2D model of hyperbolic space was worked out by Henri Poincare. The lines are called Poincare lines.*

Here are some of the consequences of that:

- The sum of the angles are always less than 180 degrees.
- Similar triangles do not exist (angular defect varies with the size of the triangles.
- Rectangles are impossible



Hyperbolic space can be thought of as the geometry of the "Saddle" shaped plane, as shown above.

20th century Dutch artist M.C. Escher use hyperbolic geometry in many of his works.

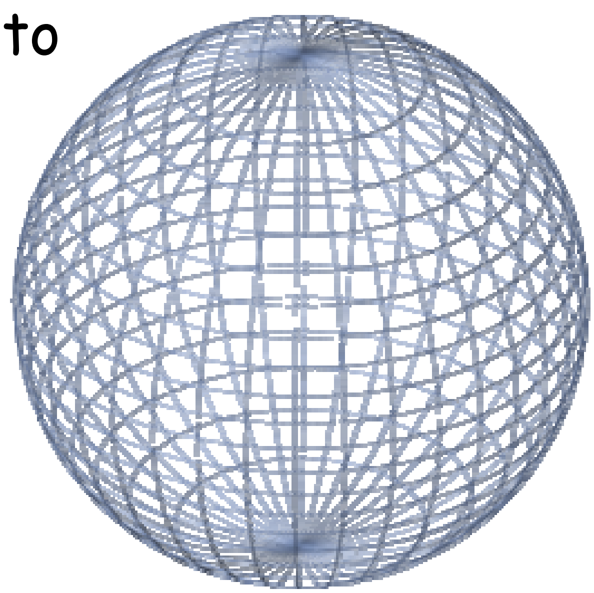
A couple of decades after the discovery of hyperbolic space, another type of non-Euclidean space was discovered: Elliptic (or Spherical) space.

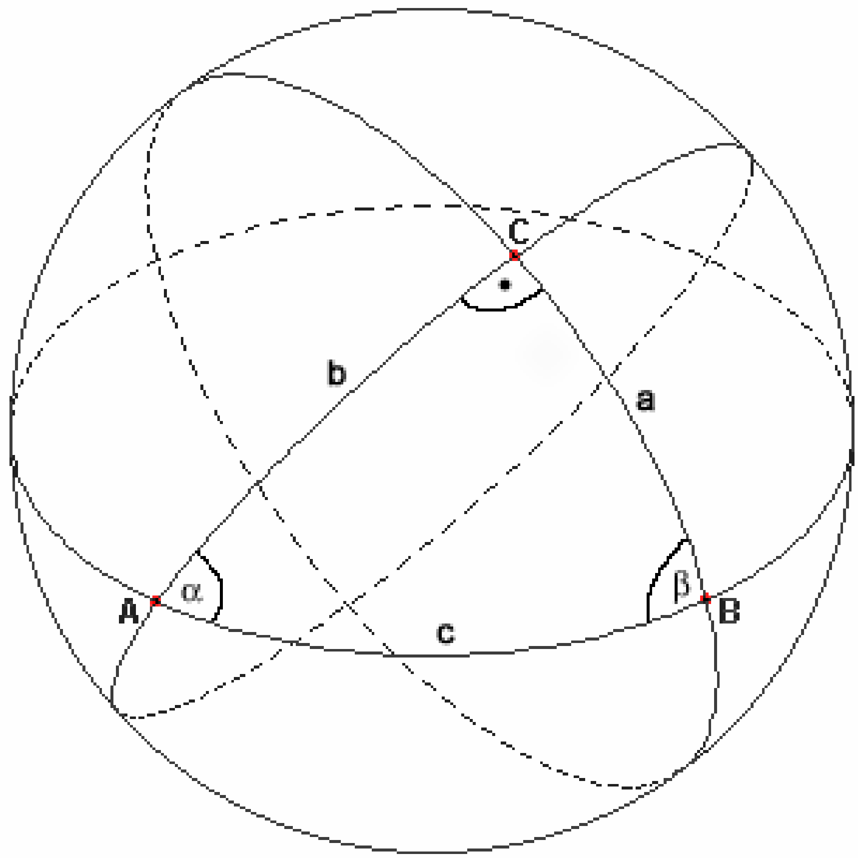
Elliptic space is the space you get if you assume the OTHER violation of the parallel postulate: that NO parallel lines exist (i.e. all lines in the plane must intersect).

In two dimensions, this type of space was known and studied in a different context by the Greeks, and even by Gauss, yet they didn't grasp its significance as an example of an elliptic space.

So what does Elliptic Geometry look like? What are its implications?

The best way to picture this is to look at a globe. The lines in elliptic space are great circles, and are the longitudinal lines of a globe. They are exactly next to each other, yet they eventually intersect.



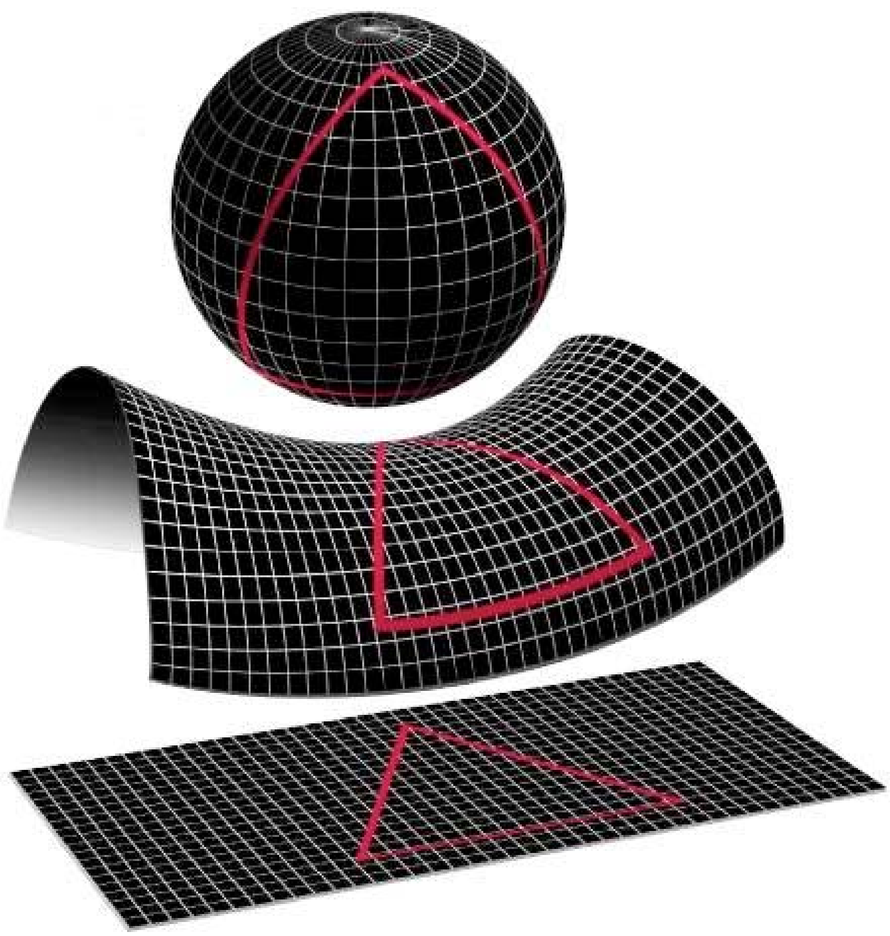



- A triangle is formed by the intersection of three lines (great circles). The sum of the angles is GREATER than 180 degrees!

It was again Herr Gauss who led the mathematical community in the study of elliptic geometry, but it was one of his students, Georg Bernhard Riemann who fueled the curved-space revolution of elliptic geometry, which culminated with Einstein's theory of relativity in 1905.

# Say What?!?!

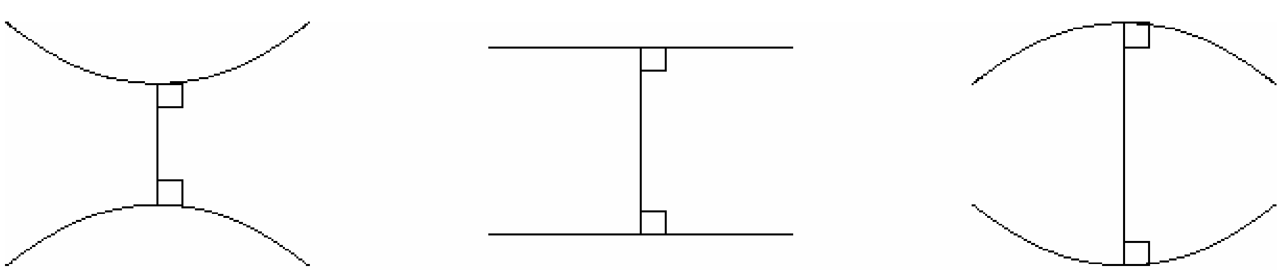
So we now have 3 separate geometries based on the interpretation of Euclid's 5th postulate, or the parallel postulate. We can summarize them here.



Elliptic Space:
Positive Curvature

Hyperbolic Space:
Negative Curvature

Flat/Euclidean Space:
No Curvature



Hyperbolic   Euclidean   Elliptic

# Recourses:

Images from
www.wikipedia.com

Information from
**Euclid's Window: The Story of Geometry from Parallel Lines to Hyperspace**
By Leonard Mlodinow